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COMMENT

On the Lie symmetries of the 2D Lotka–Volterra system

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Abstract. We show that the two-dimensional Lotka–Volterra system also has a non-trivial symmetry group.

The Lie symmetries of the two-dimensional Lotka–Volterra system

$$x' = ax - xy \qquad y' = xy - by \tag{1}$$

has been rediscussed recently by Lakshmanan and Senthil Velan in [1]. Baumann and Freyberg [2] showed that equation (1) admits a three-parameter symmetry group (up to the quadratic power in the variable y) provided the parametric condition, $a + b = 0$, is satisfied. Lakshmanan and Senthil Velan found that, for the same parametric condition, we can find an infinite sequence of symmetries, not functionally independent, and a related infinite-dimensional Lie algebra. From these symmetries we can identify integrals of motion for system (1), if $a + b = 0$.

We note here that equation (1), in the general case, also admits a non-trivial Lie symmetry. If we apply the Lie conditions [1, 3] to this system we get the following equations to be satisfied:

$$\begin{aligned} &(y-a)\eta_1 + x\eta_2 + \partial_t\eta_1 + (ax-xy)\partial_x\eta_1 + (xy-by)\partial_y\eta_1 = 0 \\ &-y\eta_1 + (b-x)\eta_2 + \partial_t\eta_2 + (ax-xy)\partial_x\eta_2 + (xy-by)\partial_y\eta_2 = 0\,. \end{aligned} \tag{2}$$

Solving this system, with the supposition that η_1 and η_2 are time-independent functions, we get the symmetry vector fields

$$\begin{aligned} &U_1 = (ax-xy)\partial_x + (xy-by)\partial_y \\ &U_2 = x^b y^a \mathrm{e}^{-(x+y)} U_1\,. \end{aligned} \tag{3}$$

The well known first integral $I = x^b y^a \mathrm{e}^{-(x+y)}$ is found immediately.

We have here a simple example of a Lie symmetry vector field with a transcendental dependence on the variables. Usually this kind of symmetry is not found due to the supposition of a polynomial dependence of the symmetry on the variables of the system. For the 3D Lotka–Volterra system we can make a similar analysis and find a lot of transcendental symmetries and related first integrals [4].

References

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